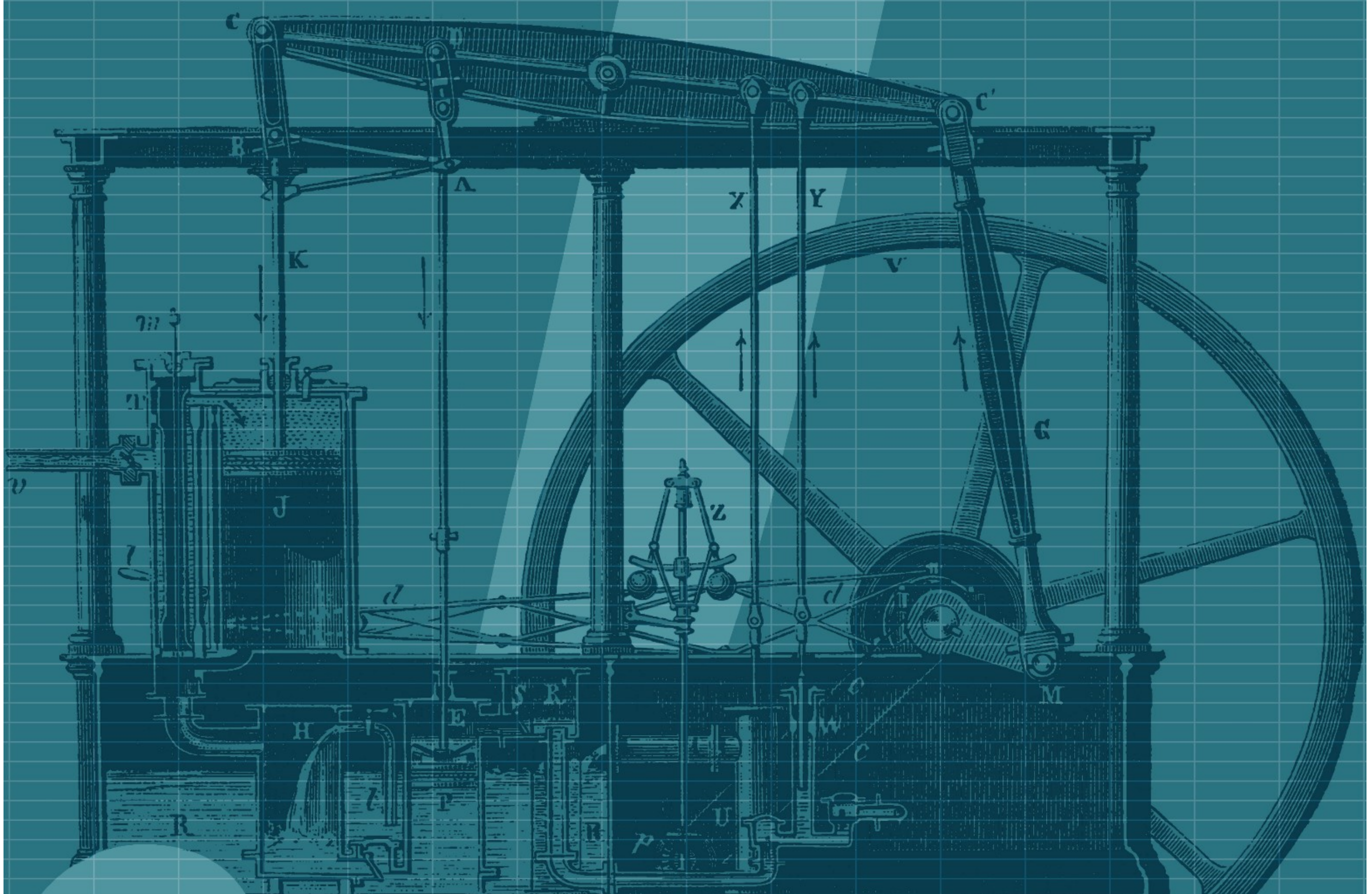
WHAT IF... THE IDEA THAT CHANGED THE WORLD

It has been called the most important concept in mathematics,
now function can enable every student to learn math as a creative experience.



ART BARDIGE

It has been called the most important concept in mathematics. It birthed the scientific revolution. It enabled computer programming. In the latest incarnation, spreadsheets, it revolutionized business. Yet few of us can name it; still fewer of us can define it; and it is unlikely even a handful of us know its evolution. This idea that changed the world will now enable every student to learn mathematics as a creative experience.

Preface

The math students are required to master today was designed and developed for medieval business in the year 1202. It is no longer used in business. It is obsolete!

This white paper describes a reinvention of mathematics education asking:

- "What if we developed a new math curriculum and pedagogy designed to prepare students for 21st century business?"
- "What if spreadsheets the ubiquitous business quantitative tool were, along with the Internet, always available for students?"
- "What if we wanted every student to learn math as a creative experience?"

Cover

The image on the cover is a drawing of James Watt's first fully functioning steam engine. Like all machines it has inputs, outputs, and actions (rules) that connect them. In the image you can trace the steam input, the wheel rotation output, and all of the input-output connections between them. In the center of the image, the inverted V is the governor which is a rule to control the engine's speed by taking feedback from the output and using it to control the input. Spreadsheets are function machines based on the same basic principles as Watt's steam engine.

Cover Design by Ryan McQuade

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What if Math, *www.whatifmath.org, is a project of Sustainablearning a 501(c)(3). Available free to students and teachers, its hundred+ Problem-Based-Learning lessons use spreadsheets as math laboratories to ask "What if…" and not "What is____?*

My biography can be found on my personal website www.artifacts.com.

This paper has been enriched by many including my partners at What if Math: Peter Mili, Ryan McQuade, Steve Bayle, my kids Kori, Brenan, Arran, and editing of my wife Betty and Frank Ferguson.

Cover Image – Engraving of a James Watt early steam engine 1784

What if Math (whatifmath.org)

Contents

WHAT IF…

THE IDEA THAT CHANGED THE WORLD

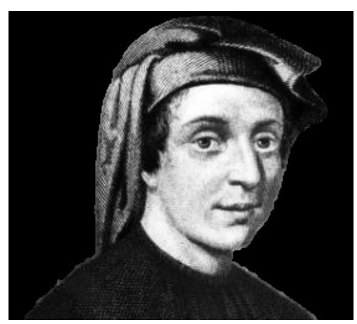
ART BARDIGE

Leonardo's Math

Some profound ideas like natural selection, general relativity, or calculus arrive at the intellectual world stage full-blown in the works of one or perhaps two people. Some like quantum mechanics may take a generation or more to develop. And some like algebra or geometry, despite their long history, could be said to be founded in a single great synthesizing document. But function is a rare bird, its origins obscured inside the general development of the algebra of equations and its importance overwhelmed by the great invention of the calculus. However, the story of its birth is worth the telling, for this rarest of inventions in the whole history of ideas had its origins in the virtually simultaneous publication of disparate works by likely unrelated authors between 1635 to 1639. These works help us illuminate this profound idea, understand its importance, and clearly recognize the role it must now play in mathematics education.

To place these works in context, we must start at the beginning of the algebra revolution, the 9th century work of al Khwarizmi, a Persian Muslim scholar in Baghdad's House of Wisdom and its founding document *The Compendious Book of Calculation by Completion and Balancin*g written around 820.[i] Our word algebra comes from its Arabic title. Its methods and standard processes for completing and balancing equations to solve for an unknown quantity which we would call algorithms, a corruption of his name, are familiar to algebra students today. Neither the problems nor the processes were represented symbolically, for other than the Arabic numerals themselves and a few abbreviations for operations, arithmetic and algebra were verbal languages and would generally remain so for nearly 800 years.

Algebra and algorithms entered European commerce in 1202 when Leonardo of Pisa published his great work, *Liber abbaci (The Book of Calculation)*. Born in Pisa about the year the foundation was laid for the campanile, the bell tower soon to gain the nickname The Leaning Tower, Leonardo soon went to Algeria to join his trader and ambassador father, who had him tutored in Arabic arithmetic and algebra. These "academic subjects" were not used in commerce or trade, for Roman math with its alphabet-based tally system and abacus for computation was ubiquitous. Becoming a merchant himself, Leonardo fully recognized the problems the archaic Roman system caused. While good enough for the Empire with its single coinage and weights and measures that required traders to mainly



[i] Some scholars attribute the founding of algebra to one, Diophantus, a shadowy figure historians believed lived in Alexandria in the 3rd century AD. Though the surviving chapters of his *Arithmetica* do not have generalized methods we think of as algebra, they did include generalized problems like solving certain quadratic equations.

add and subtract, Roman math made computation difficult in medieval times when most cities and towns had their own weights, measures, and monetary systems. Products like spice, gems, furs, and cloth traveled long distances and passed through many hands requiring constant re-measure and re-computing. And each calculation involved ratio and proportion reasoning requiring multiplication and division operations, and two-step and more problem solving reasoning.[ii]

In 1200, Leonardo headed back to Pisa to write and publish his book to provide Pisans and other European merchants new tools for their business. He introduced them to Arabic numerals which came originally from India, place value, the algorithms to compute with both whole numbers and fractions, and this new algebra to solve equations for unknown quantities of the kind merchants constantly faced. *Liber abbaci's*[1] 15 chapters are filled with problems merchants and traders might face.

1. *On the recognition of the nine Indian figures and how all numbers are written with them. (place value)*[iii]
2. *On the multiplication of whole numbers*
3. *On the addition of them, one to another*
4. *On the subtraction of lesser numbers from greater numbers*
5. *On the division of integral numbers*
6. *On the multiplication of integral numbers with fractions*
7. *On the addition and subtraction and division of numbers and fractions and the reduction of parts*
8. *On the buying and selling of commercial things (ratio & proportion)*
9. *On the barter of commercial things (rate)*
10. *On companies made among parties (percents)*
11. *On the alloying of money (mixture problems)*
12. *On the solutions of many problems (Fibonacci sequence)*
13. *On the rule of elchataym*[2] *by which problems of false position are solved. (solving linear equations)*
14. *On the finding of square and cube roots, on binomials and their roots.*
15. *On the pertinent rules of geometric proportions*

[ii] Where we have to solve one problem by using values from another. Two-step problem solving is typical of the kind of problems students have great difficulty with starting in 5th grade.
[iii] Since chapter titles were added later or taken from first sentence of the chapter, I have added the words in parentheses describing in modern terms their general content.

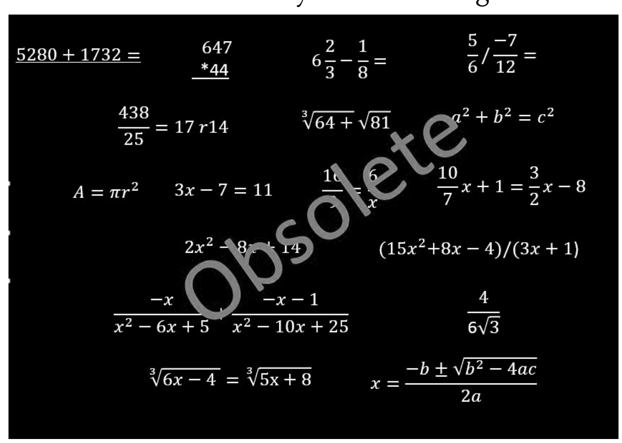
This table of contents changed my life. For I recognized it as the list of topics, in virtually the same order, that we teach our children in the K-12 math curriculum. Like us he starts with place value and goes on to whole number operations,[iv] then to whole number by fraction operations, and then fraction by fraction operations. Leonardo's math does not include decimals because they were invented almost 400 years later. Though the titles of the next chapters tend to obscure the concepts, a quick perusal of the contents in *Liber abbaci* enables us to see that they deal with ratio, rate, proportion, and something like percentage. The last chapters follow our standard algebra syllabus solving linear equations, quadratic, cubic equations, and finally a little geometry.

Algorist vs. abacist

I was awestruck. I realized that the math we teach our children is not basic or foundational at all. It is conceptually and fundamentally the math designed by Leonardo for 13th century merchants.

Over the next four centuries Leonardo's math became symbolic and the standard for business across both Europe and the world. By the early 1500 the goddess of mathematics has clearly made her choice. Today, this canonical mathematics curriculum, reified by the Common Core Standards, is the math every student must master to complete K-12.

Revolutions, even intellectual revolutions, are usually initiated or accelerated by new technologies and Leonardo's using the new technology of cheap plentiful paper, was no exception. Originally imported from China in the 12th century, paper was expensive, colored, and used for decoration. European waterpower soon made it plentiful and cheap replacing papyrus and animal skins as writing surfaces. Arithmetic and algebraic paper-based algorithms became the means for computation, problem solving, and bookkeeping. By the early 17th century Leonardo's math was fully symbolic with most of the symbols, algorithms, and concepts our children still learn today. Seventeenth century efficient computational algorithms, developed for counting houses, survive in our classrooms. Yet around this very time, titanic changes in mathematics were occurring that now make Leonardo's arithmetic and algebra with its paper algorithms obsolete.

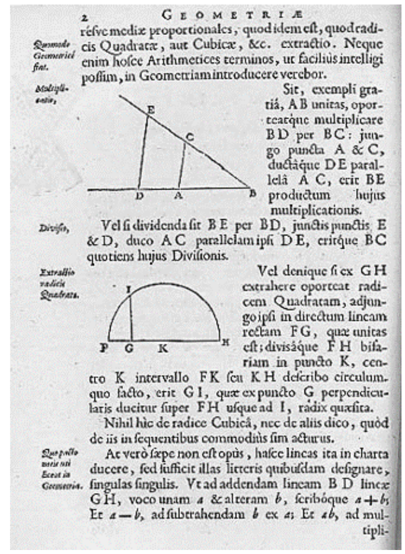


[iv] Leonardo's multiplication before addition makes sense to me. The concept of multiplication as counting-by's or even as adding the same number again and again is easier than the addition of two different numbers, "counting-on." I believe the shift from starting with multiplication to starting with addition came with the development of the multiplication algorithm which is certainly more complex that the addition algorithm.

Functional Thinking

The wonder of it has to be the timing. Within just five years, between 1635 and 1640, five men, living in different places, with little connection or mobility, each building from his own foundation and focus, produced a revolutionary way of thinking that transformed mathematics and the sciences, fostering our modern world. Each grappled with the concept of variable and with the relationships between variables which later defined functions.

- **1637**: Rene Descartes, living in the Netherlands, published *La Géométrie* as an appendix to his great philosophical treatise *Discours de la méthode* that joined algebra and geometry. He invented a new kind of quantity to represent lines and not numbers. Though he called it a new geometry because it dealt with lines and shapes, it was a new kind of algebra, an algebra of lines (collections of points) represented by variables and not points represented by numbers. In this second page of *La Géométrie* he begins to meld algebra and geometry by showing us how not only to add and subtract lines but to multiply and divide them. We may say this: before Descartes people saw lines as geometric constructions labeled by their endpoints. After Descartes they conceived of lines as representations of the relationship between two or more variables, labeled by them.


2 GEOMETRIÆ
réſve mediæ proportionales, quod idem eſt, quod radicis Quadratæ, aut Cubicæ, &c. extractio. Neque enim hoſce Arithmetices terminos, ut faciliùs intelligi poſſim, in Geometriam introducere verebor.

Quomodo Geometricè fiat.

Multiplicatio,

Sit, exempli gratiâ, AB unitas, oporteatque multiplicare BD per BC: jungo puncta A & C, ductâque DE parallelâ AC, erit BE productum hujus multiplicationis.

Diviſio,

Vel ſi dividenda ſit BE per BD, junctis punctis E & D, duco AC parallelam ipſi DE, eritque BC quotiens hujus Diviſionis.

Extractio radicis Quadratæ.

Vel denique ſi ex GH extrahere oporteat radicem Quadratam, adjungo ipſi in directum lineam rectam FG, quæ unitas eſt; diviſâque FH bifariam in puncto K, centro K intervallo FK ſeu KH deſcribo circulum. quo facto, erit GI, quæ ex puncto G perpendicularis ducitur ſuper FH uſque ad I, radix quæſita.

Nihil hìc de radice Cubicâ, nec de aliis dico, quòd de iis in ſequentibus commodiùs ſim acturus.

Quo pacto notis uti liceat in Geometria.

At verò ſæpe non eſt opùs, haſce lineas ita in charta ducere, ſed ſufficit illas litteris quibuſdam deſignare, ſingulas ſingulis. Vt ad addendam lineam BD lineæ GH, voco unam a & alteram b, ſcribóque $a+b$; Et $a-b$, ad ſubtrahendam b ex a; Et ab, ad multipli-

- **1638**: Galileo Galilei, age 74, nearly blind, and under house arrest in Florence for advocating Copernican heliocentrism, secretly sent a new manuscript to Holland for printing. It was his great work *Dialog Concerning Two New Sciences* written in Plato's Socratic style as conversations between Galileo the teacher, an Aristotelean, and a wise man. The first new science, dealt with the mechanical properties of matter. The second, still taught to every school child, was the study of motion. In its fourth and final dialog, Galileo focused on the motion of projectiles and showed that their seemingly complex path could be understood as the composition of two simpler motions, one horizontal the other vertical. The horizontal motion in the absence of air resistance was "uniform," constant with time, the vertical motion because of gravity was "accelerated" varying with time, as he had shown 40 years earlier with his inclined plane experiments. The distance covered by a falling body during each unit of time increased as the square of the time. "...if we take equal time-intervals of any size whatever, and if we imagine the particle to be carried by a similar compound motion, the positions of this particle, at the ends of these time-intervals, will lie on one and the same parabola."[3] A line can represent the motion of a body using time as a variable.[4] This amazing image of projectile motion from that dialog, the first conceptual graph[5] ever published, shows the parabola composed of horizontal and vertical motions.

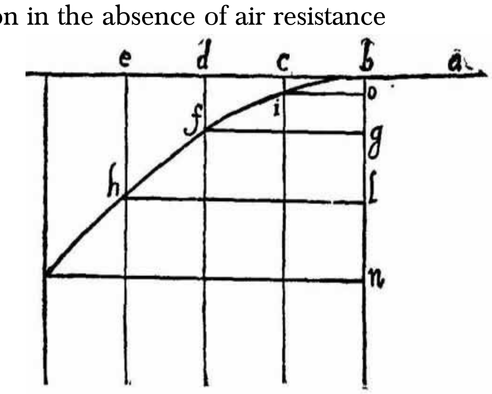
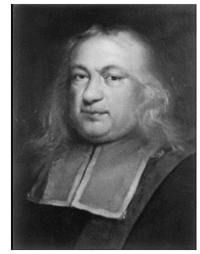


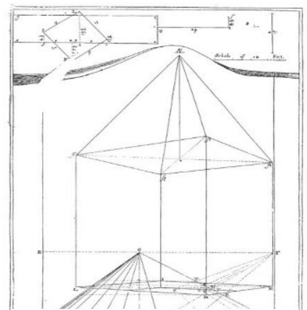
Fig. 108

- **C. 1638**: Pierre de Fermat, often remembered today for his cryptic marginal note known as “Fermat’s Last Theorem,” studied curves, developing ways to find their maxima, minima, tangents, point of inflection, and rates of change. His work, the precursor to the differential calculus of Newton and Leibniz, includes many of the topics students study in school today in courses in college algebra and precalculus. He spread these ideas from 1635-1640 through letters to friends and colleagues from his home in Toulouse, France.



- **C. 1635:** Bonaventura Cavalieri, Galileo’s protégé, studied areas and volumes and sought to find general ways to compute them. The first to find the area under a parabola, his work is considered a critical prelude to integral calculus, with his “principle of indivisibles,” imagined shapes built by composition of very small incremental slices. “Cavalieri asserted that a line was made up of an infinite number of points (each without magnitude), a surface of infinite number of lines (each without breadth), and a volume of an infinite number of surfaces (each without thickness).”[6]

- **1639:** Gerard Desargues published his study of the geometry of projected shapes introducing mathematical transformation and invariance. Though considered brilliant by his contemporaries, Desargues projective geometry did not play a significant role in mathematics until the 19th century, but represented another facet of this new way of thinking about relationships between lines (variables).



From the Brouillon Project
Geometry of an Art,

We cannot account for the amazing mathematical and scientific generativity of the early 17th century as the random arrival of individual geniuses, remarkable as they were. Nor can we ascribe it simply to a loosening of the reins of religion and other institutional structures. It must be recognized as a revolution enabled and afforded by a new and more general way of thinking, a shared way of thinking we now call functional or “What if…” thinking. It was based on a new kind of object defined by Descartes, a variable he labeled “x”, that represented a line and not just a number. That could be linked to a formula, a table of values, or a graph. It provided the sciences ways to build complex systems from simple building blocks through composition as Galileo and Desargues had. It enabled mathematicians to ask questions about change as Fermat or Cavalieri had. It would enable natural philosophers to ask the experimental science question, “What if…” It gave artists and scientists ways to visualize change as transformations. Newton described the simplicity of this wonderful new way of thinking in 1671:

> *I am amazed that it has occurred to no one…to fit the doctrine recently established for decimal numbers to variables, especially since the way is then open to more striking consequences. For since this doctrine in species has the same relationship to Algebra that the doctrine of decimal numbers has to common Arithmetic, its operations, of Addition, Subtraction, Multiplication, Division, and Root extraction may be easily learnt from the latter’s.*[7]

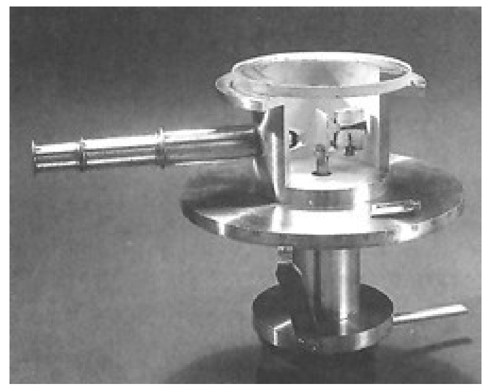
Over the next 40 years, a tenth of the time from *Liber abbaci* to *La Géométrie*, the world of mathematics and science exploded. Newton and Leibniz developed calculus with the rules for finding the rate of change of curves and the area beneath them. Leibniz applied the word functions to describe the building blocks of curves.[8] Newton applied these ideas to motion developing the first true cause-and-effect based physical theory and thus defining the course not only of physics but of the entire scientific revolution. And from this humble beginning, function has now become in the words of the Chair of Harvard University's math department:

> *Perhaps the most important concept of mathematics is that of function, which provides us with the means to study dependence and change.*[v]

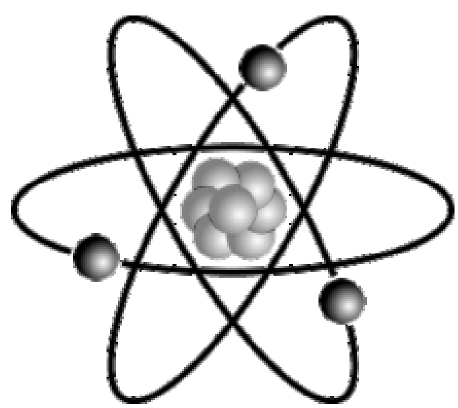
Functions and functional thinking are today at the very center of our world, pervading the disciplines, those we call STEM or STEAM as well as those we label the "Liberal Arts" to include the whole of the humanities.

Science

Galileo not only helped invent functional thinking, he applied it, inventing experimental science with his study of motion on inclined planes. Rolling balls down troughs he set at various angles and measuring the distance the ball traveled in units of time. By asking "What if…" he diluted gravity and found its acceleration to be constant and independent of the weight of the objects he rolled or objects he would drop. This first scientific experiment asking "What if…" remains the archetype for all of science. We can see it in the same way in another canonical experiment. In 1911 Ernest Rutherford aimed a source of alpha particles (the nuclei of helium) toward a thin sheet of gold and from there to a scintillation screen that glowed when hit by these particles. Looking for the structure of atoms, Rutherford had his assistant Hans Geiger sit in a dark room staring for long hours at that screen, counting dots of light and measuring the distance they were deflected from center. He found that most alpha particles went through the gold foil as if it were not there, with a few deviating slightly. When Geiger wanted to add Ernest Marsden to relieve himself of that boring task, Rutherford had a brainstorm, "Why not let him see whether any α particles can be scattered through a large angle?" What if some of the particles bounced straight back?



Marsden found that some actually did! A very small number of alpha particles came back toward the source. From this experimental result Rutherford imagined a new model of the atom, the familiar atom we picture as the atomic energy logo. And using the deflection measures he was able to actually calculate the size of the nucleus and of the atom itself. The atom is mostly empty space, much more so than is shown in this standard picture, with its nucleus is just 1/10,000th its size! It therefore looks nothing like this standard image most of us hold. Asking What if… in experiments leads us to build models of our world, models that in turn enable us to both explain and predict.



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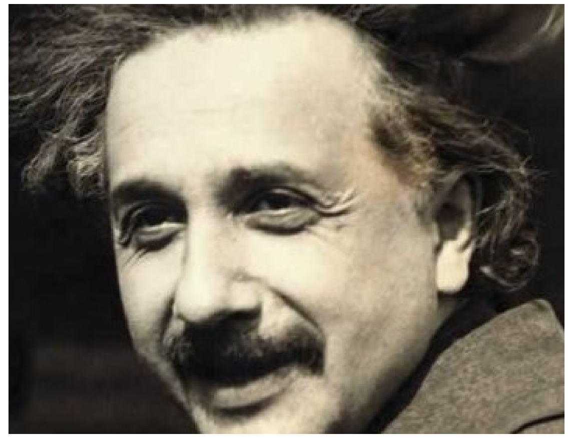
[v] Peter Kronheimer,[v] posted in the Harvard student handbook:

Experiments, the heart of science, are "What if…" questions. Scientists use experiments to build and test models searching for patterns. They think functionally, asking: "What if I change the input?" "What if the output were…? "What if I suggest a new rule governing that process?" "What if…" questions connect causes to effects and rules to models, laws, or theories. Einstein asked: "What if I rode on a light beam?" "What if I wanted to synchronize clocks with someone riding a train?" "What if I were in an elevator far out in space?" The "What if…" world of functional thinking is the world of the science laboratory both physical and mental. It is the world of experiment and theory.



Technology

I had the good fortune to code when personal computing was in its infancy and microprocessors were much simpler. That simplicity was quite apparent in my Apple II microprocessor called 6502. Though BASIC was the language of choice for the Apple II, it was not fast enough for the things I wanted to program, so I turned to Assembly Language, the collection of three letter codes that represented the instructions built into the microprocessor. The 6502 had only about 50 instructions, primitive by today's standards, that can be divided into essentially three groups:

- **Housekeeping** commands included things like: LDA and STA to load and store data to special locations or to main memory, or keep track of actions with counters like INC or DEC (increment or decrement).
- **Arithmetic** commands included things like ADC and SBC to add, subtract, and deal with carry. Note there is no multiply or divide, for these operations were done by repeated addition or subtraction.
- **Logical** commands, the tools that make computers different from calculators, like BCC, CMP, or JMP to branch, compare, and jump to a subroutine build on the general "if…then" form, supported by AND or ORA for "and" and "or" logic.

ADC	add with carry
AND	and (with accumulator)
ASL	arithmetic shift left
BCC	branch on carry clear
BCS	branch on carry set
BEQ	branch on equal (zero set)
BIT	bit test
BMI	branch on minus (negative set)
BNE	branch on not equal (zero clear)
BPL	branch on plus (negative clear)
BRK	interrupt
BVC	branch on overflow clear
BVS	branch on overflow set
CLC	clear carry
CLD	clear decimal
CLI	clear interrupt disable
CLV	clear overflow
CMP	compare (with accumulator)
CPX	compare with X
CPY	compare with Y
DEC	decrement
DEX	decrement X
DEY	decrement Y
EOR	exclusive or (with accumulator)
INC	increment
INX	increment X
INY	increment Y
JMP	jump
JSR	jump subroutine
LDA	load accumulator
LDY	load X
LDY	load Y
LSR	logical shift right
NOP	no operation
ORA	or with accumulator
PHA	push accumulator
PHP	push processor status (SR)
PLA	pull accumulator
PLP	pull processor status (SR)
ROL	rotate left
ROR	rotate right
RTI	return from interrupt
RTS	return from subroutine
SBC	subtract with carry
SEC	set carry
SED	set decimal
SEI	set interrupt disable
STA	store accumulator
STX	store X
STY	store Y
TAX	transfer accumulator to X
TAY	transfer accumulator to Y
TSX	transfer stack pointer to X
TXA	transfer X to accumulator
TXS	transfer X to stack pointer
TYA	transfer Y to accumulator

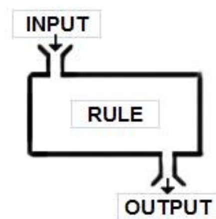
Technology and coding at their very heart are functional, "What if…," thinking. Spreadsheets and spreadsheet-based lessons remind us of the wonderful power coding can have on our minds; the deep desire to make the code work; to see, literally see, a model function; to feel the exhilaration of bringing imagination to life. Coding, with its creative capability, (and spreadsheets are coding platforms), can engage every student in ways we just dream of today. For coding is functional thinking. As Ada Lovelace, credited with inventing programming, taught nearly two centuries ago:

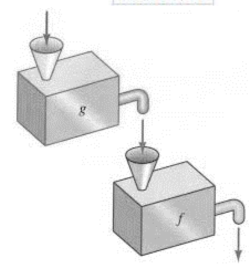
> *"developping [sic] and tabulating any function whatever. . . the engine [is] the material expression of any indefinite function of any degree of generality and complexity.*[9]

Engineering

The term "mechanics" describes that arena of physics devoted to bodies in motion. Today in physics we talk about "classical mechanics", "quantum mechanics", "relativistic mechanics" and more. Mechanics is an old name that came from visualizing the universe as a great machine, a clockwork, with moving objects interacting. It is a symbol of the close association of engineering and the physical sciences. Much of physics and our physical intuition rests on this connection between machines and our concepts of nature. Einstein's long stint as a patent clerk certainly built his incredible physical intuition, for learning to invent and describe new machines through concrete visualizations is a skill so often required to ground abstract conceptualizations.

Engineering is built on "What if…" functional thinking. Whether designing new bridges or inventing new phones, engineers spend their days asking "What if…" "What if this needs to support more weight?" "What if this part fails?" "What if I change this or make this smaller or make a car without a driver?" All machines have some kind of input and produce some kind output. They follow rules and engineers build models and test those models with "What if…" questions. All machines are essentially functions that transform inputs into outputs. And when we think of functions, we can picture a machine.





The standard picture of function, even used with young students, has an input funnel, an output funnel, and a rule that resides in the box linking them.[10] We ask them to imagine a machine inside that box performing some "mechanical" task like adding 3 to any input number, and we ask them, "What will the output be?" Or we play "guess my rule" when given both an input and an output we ask what rule will connect them. We even give them the output and the rule and ask what the input will have to be. This concrete image helps make even the most abstract function easier to understand, work with, and apply. This mechanical basis of functional thinking lends it a great power to build models concretely, like this one which shows the output of one function being the input of another function. Mathematics, even powerful mathematics, need not be just symbolic abstractions.

Art

What if… functional thinking is not limited to STEM disciplines, for it is at the very heart of the arts and humanities as well. When we create a work of art we always ask: "What if… I change this color, transform this shape, use this key, explore this metaphor, play this faster, work in this style." When I use the term art, I envision the whole of the humanities. Poetry is just as much a "What if…" arena as science is. For poets teach us to make patterns and often through those patterns give us the vocabulary and even the grammar to build other patterns. Perhaps the most famous example is the "quark", the invention of Murray Gell-Mann in 1964, who took this name for his new fundamental constituent of nature from James Joyce's *Finnegans Wake*, "Three quarks for Muster Mark!" Though George Zweig came up with the same concept independently, he called the new particles that made up protons and neutrons "aces." The symmetry embodied in Joyce's poetic line and in Gell-Mann's choice of words was emblematic of the symmetry in the theory, and Gell-Mann's name stuck.

We are all certainly artists. I find that each of us has an art, at least one, that we love and enjoy. Mine is photography. I have loved taking pictures ever since I was a young boy. As soon as I could afford it, I bought a single lens reflex camera. I did a darkroom stint developing, printing, and enlarging black and white photos, some of which still compete in my home with the works of great professional photographers. But it was not until I started using digital photography that I was ultimately smitten. I began to take photos everywhere, enhance them in Picasa, print them in 8 ½x11 or 13x19, encase them in plastic for placemats or mount them on foam-core for walls.

I would play with each photo in Picasa, transforming it from an okay image, the input, to a work of art, the output, asking "What if…" I change the contrast, saturate the color, crop the image, add more blue to the sky, or turn it orange. "What if…" I darkened the background or made it brighter, turned it to sepia, brought out one of the colors, or added shadows? The possibilities seemed to be endless, and this simple program enabled me to do so much "What if…" thinking that it was sufficient for me. It was the technology that made the difference, it enabled me to easily and quickly experiment, to do and undo, to keep track of my rules, and to start all over when I got into trouble. The technology liberated the artist in me and enabled me to play, to experiment, to be creative and to share my work with others.

Enabling every student to be an artist as they learn mathematics would transform what for many is a dull feared subject to a loved one. It could make mathematics for every student a creative subject, and I believe there is great power in this human drive. As I found in photography, the technology can liberate our imaginations and engage us all in creative patternmaking to make the learning of mathematics a creative experience.

Math

Now that we have laid the foundation for functional thinking in history and example, we are ready to ask, "What if we apply functional thinking to math education?" Without question mathematics has become increasingly important in education. It is widely acknowledged that the good jobs of the future will be STEM jobs and even those students not directly connected to that world will still be heavily impacted by it, for mathematics and mathematical thinking is rapidly becoming as important as verbal literacy for a productive life. We must further acknowledge that our students' success in math education have not substantially improved over the past 50 years. We have a measure of this failure in the results of the gold standards NAEP[11] (National Assessment of Educational Progress) test. During that time, we have tripled the amount we spend in real dollars per student, reduced class size, promoted professional development of teachers and have tried literally hundreds of broad experiments with new curriculum, school structures, and new pedagogy. We have sought through new standards, advertising, and testing to motivate students to learn it. Yet we have seen no improvement. Our graph is flatline. Its slope is zero. Zero can be a very large number![vi]

[vi] The percentage of students requiring remedial college math has remained around 36% or above for at least two decades. Most of the state and national tests of the critical 8th grade students have at least half scoring below

If learning Leonardo's math were easy or quick for most students, then we should expect an 85% success pass rate, the common standard. But by any reasonable estimate, though we have really tried for several generations, our mathematics education success rate remains at no more than 20%[12].

The math is perfectly clear. Our children spend between 200 and 300 hours per year on average studying math inside and outside classrooms for 12 or more years, the equivalent of 2 man-years of work, and most are failing to learn the subject well enough to use it and get the education and jobs of their dreams. We don't know why so many of our children find, even what is actually a rather small amount of math, so hard to learn. We don't know why some students get math and most don't. We don't really know why some populations like math and most don't.

But we do know that much of the math we are asking all of our children to master is obsolete. This is math that will never be used outside of school. They will not use it in their real lives. They do not gain understanding from it. They don't need it!

proficient for the past two decades with no substantial improvement over this same period. By any measure we have failed to significantly improve math learning despite the effort and attention given the problem.

What if… We Reinvented Mathematics Education?

Since "What if…" functional thinking is now the basis for our experimental sciences, for engineering, for technology, for the arts, of course for mathematics, and as we shall soon see for business today, then we are required to ask today's essential math education question:

> *"What if we were to reinvent our mathematics curriculum based on the needs of students for work and life in the 21st century?"*

Before we look at the new elements we should add to the curriculum, we need to make room. We need to ask, "What should we subtract?" [vii] "Would a 21st century math curriculum include:"

- **Long division?** The square root paper algorithm vanished from K-12 education in America 50 years ago. Today, isn't the paper algorithm for division (long division) just as unnecessary for students to master? Do they really need to learn to fluently and accurately add long columns of numbers on paper, to regroup or "borrow" to subtract, or to multiply 3 and 4 digit numbers on paper? We are not talking here about concepts, about understanding the operations. That understanding is essential to any math curriculum. Nor are we talking about memorizing the times table so we can calculate mentally. But the paper algorithms that our children spend much of their math school time doing and redoing, paper worksheet after paper worksheet to "practice" whole numbers, fractions, and decimals algorithms, are any of them really necessary anymore? Are our children going to need them when even today's most basic cell phones have built-in calculators? And even if all students could build this fluency today, if it were easy and quick for all to learn, wouldn't they lose it anyway? Maintaining any fluency requires practice and as adults they will most certainly never use or practice paper algorithms. Do any of the paper math exercises our children endlessly repeat today have real value?
- **Leonardo's algebra?** Most of the algebra our students are required to master involves manipulating expressions and solving equations. These procedures or algorithms for factoring or balancing equations are designed for paper and carefully chosen to be paper computable. Do we need to teach our children to complete the square to solve peculiar quadratic equations that happen to fit that model? When paper tools were the only means to solve equations special algorithms and tricks were necessary. But type an equation into Google's search box, and it will instantly produce a graph so you can visualize its solutions or solve it using any number of free sites in Google's list. Why-oh-why do students need to master paper algorithms for the special cases? They think it abstract and can't picture applying it to the real world. And they feel that they are only practicing moving meaningless symbols around to follow arcane rules and asking, "Why am I learning this?"

[vii] The Common Core, in its effort to add the conceptual to procedural and accelerating the curriculum with algebra for all in 8th grade, has placed heavy demands on students and teachers. Though they claim to have streamlined the curriculum by taking out the repetition, that only increased the demands on students who are less prepared, who have fallen behind, or who find math difficult to learn. If we are to learn the lessons of this effort to reinvent mathematics education, we must conclude that our first order of business is to start with a radically smaller set of preexisting demands so that we can add new and exciting ideas and activities without overloading the system.

- **Leonardo's sequence?** Taking out the paper algorithms not only takes a big chunk out of our current curriculum, it also removes the impediment to applying mathematics to authentic problem solving. Today's math curriculum is a staircase,[viii] a well-defined sequence of topics that generally follow Leonardo's syllabus. This sequence is based on algorithmic difficulty not conceptual difficulty. Conceptually multiplication (counting-by a common number) is not fundamentally harder than addition. Subtraction is conceptually more difficult than division, for even very young children can deal out cards to multiple players, but ask them to explain subtraction on their fingers, well that is a very different story.[ix] Eliminating the algorithmic staircase frees us from the constraints that shaped our rigid grade-based schools, made personalization and individualization extremely difficult, and prevent us from broadly integrating disciplines.

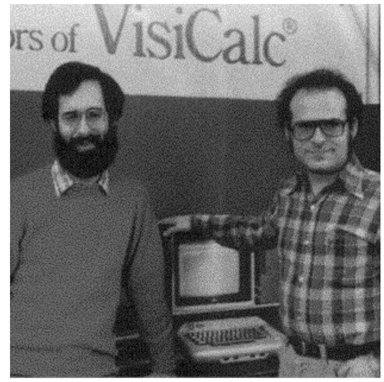
Common Core Staircase

- **Headmath as well as Handmath?** For many, a math program based on technology raises serious concerns that calculators and computers take a big toll on numbersense. They complain that people coming into the workforce today cannot solve problems in their heads, can't approximate, give a quick "ballpark answer," see an order of magnitude, or perform a seat-of-the-pants calculation. And they blame it on the use of technology and the lack of paper practice. They feel that paper practice builds conceptual understanding and is the key to "getting math". Their numbersense concerns are, no doubt, a big problem. We need everyone to know if a number makes sense, to be able to mentally check machine computations, and to quickly recognize if something is affordable or just plain right. But does this capability stem from practicing the paper algorithms? I don't think so. We actually teach our children two very different forms of mathematics, *handmath* that we do with our fingers with pencils on paper, on calculators, or even on computers, and *headmath* that we do in our minds. Handmath is algorithmic, procedures that are carefully laid out, followed, and practiced. These procedures produce "correct" answers. Headmath uses simple patterns, shortcuts, even tricks to get a sense of the answer. Headmath is built on strong numbersense and patternmaking. When we do headmath we do not carry to add or borrow to subtract, sum partial products to multiply or multiply and take differences to divide. We estimate and approximate, move digits around, cross multiply, and change them to get close for the "reality check". It is not done on paper, it cannot be easily tested on paper, and it has thus evaporated from our test-reliant curriculum. When calculators made slide rules obsolete, the order of magnitude exercises of generations of science students were no longer practiced. **Weak mental numbersense is not due to the use of calculators or a paucity of paper-computational practice. It is due to a lack of headmath practice that we should and must return to our curriculum.**

[viii] This picture of the Common Core staircase is taken from their website.

[ix] Show 5 fingers on your left hand and two on your right hand. Then 5 take away 2 is…? Looks to me that the 5 are still there when we take away the 2. Of course that is not what we meant but what do young children see?

The Mathematics of Business Today



Leonardo's math is obsolete! Based on paper algorithms, it belongs in museums and not in classrooms. Now that we have made room in our math curriculum, what do we replace Leonardo's math with? Since Leonardo of Pisa based his mathematics curriculum on the needs of business in the 13^{th} century should we not base our math curriculum and pedagogy on the needs of business in 21^{st} century. Unlike the merchants and traders in medieval times, business today is broad and complex. It ranges from managing personal finances to running large corporations, from building a new app to developing major scientific research projects, from planning a classroom assignment to manufacturing the next generation of automobiles, from individual consulting to overseeing large government agencies. Yet even when we think about business in the broadest sense of the word, we can find common ground in the kinds of problems faced and the quantitative means used to solve them.

The revolution in the mathematics of business that ended Leonardo's math reign began in the year 1979. The new personal computer provided the breakthrough technology for the invention of the spreadsheet by Bob Frankston and Dan Bricklin. Though at first they thought of *VisiCalc* as a visual calculator; they designed it in the genre of computer programming language—as a function machine with inputs, outputs, and rules (formulas)—and soon came to recognize its "What if…" power.

Before spreadsheets, a company planning a project or product would build a business model to calculate its cost and return on investment. Such models were often contracted to big accounting/consulting firms where paper worksheet models were laid out on long conference tables by taping together standard 11 x 14 sheets of accounting paper and entering numbers using #2 pencils. Pencil bits and pink eraser curls soon littered the floor. Rolls and rolls of two inch adding machine paper-tape stapled to a row or column verified the arithmetic and dangled off the table. Presenting such models to the client must have been a terrifying test for the accountants and developers who created these models, for they generally prompted the "What if…" questions from the managers; "What if…we change the interest rate by half a point, or the timeframe by 3 months, or the pricing by 7%?" The erasers and pencils and paper tapes would fly and the billing would grow. Executives quickly learned not to ask "What if…" very often.

> *"The spreadsheet enabled an executive to ask "What if…" as many times as she or he wanted, profoundly changing the way business operated.* [x] *"*

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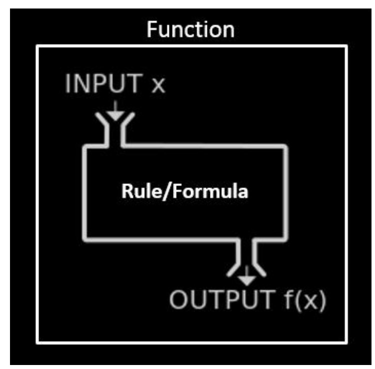
[x] Allen Sneider, an accountant and one of the earliest adopters of spreadsheets. This story came from a personal conversation with Allen.

The spreadsheet enabled individual entrepreneurs, executives, and managers to build their own models and to ask "What if…" as often as they wanted. It was likely a primary cause of the entrepreneurial revolution that began in the 1980's, built on spreadsheet generated business plans and venture capital. Soon after the spreadsheet's invention, Mitch Kapor, seeing the need for visualizing this new wealth of VisiCalc data, developed *Lotus 123,* integrating spreadsheets with graphs and database tools for the new business-oriented IBM PC's. The spreadsheet and the personal computer became a requirement for every business and a full realization of functional thinking. Excel, originally built for the new Mac in 1984, with its simplified interface, became and remains the standard quantitative data tool for all business and for most of science, technology, and engineering. Today, the spreadsheet is the ubiquitous quantitative tool for business. It is a software offering from Microsoft, Apple, and Google. It is on every desktop, tablet, and smartphone. It is available on most every student computer, usually at no cost. Though a calculating device, list manager, and means to create pretty charts, the great power of the spreadsheet is in its capacity to engage students in "What if…" creative thinking.

Spreadsheets are Function Machines



Spreadsheets are function machines with inputs, outputs, and rules. An input cell can contain a number, word, or the address of another cell. An output cell has a rule, and the spreadsheet knows it is a rule because it starts with an **=** sign[xi]. Though spreadsheets do not use variables like **x,** they use cell addresses and tables for the same purpose

Function is powerful. It not only quantifies cause and effect, but it can act as a new object. Like a number, a function can be the input to a function. So we can perform operations on functions as if they were quantities and build complex functions from simple ones. We can make the output of one function the input to another one or even the input to itself (recursion). We can add, subtract, multiply, and divide functions, and even take a function of a function (composition). We can make functions with one variable or many. they are an incredibly flexible tool. Spreadsheets call the rule of a function, a formula and collections of functions, a *model*; store them in *libraries* and use them as a coder would use *subroutines* with labels like **=SUM()**[xii].

Spreadsheets enable us to copy and apply functions intelligently. A function is not just the rule itself, it includes all of its possible inputs which we call its *domain* and all of its possible outputs, which we call its *range*.[13] Spreadsheets enable us to use copy and paste on functions and not just on their rules. Copying and pasting a rule thus normally includes the cells it expects to be its domain and range. This so-called *relative addressing* means that a rule copied from one cell to another or into a column or row would use the appropriate inputs. A rule can also be tied to a particular cell, a fixed *absolute address,* by starting the address with a $ sign. This cut and paste power with relative and absolute addressing lets us create

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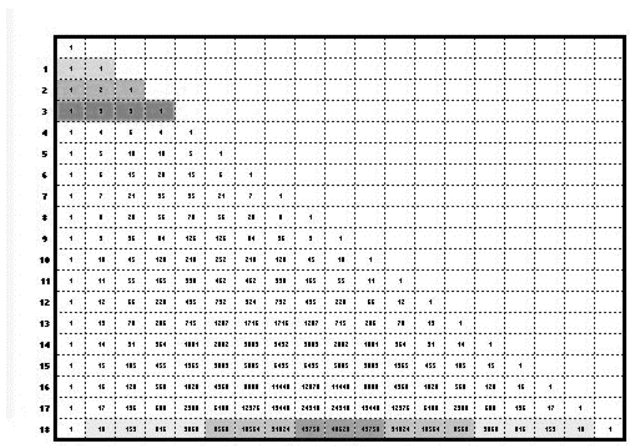
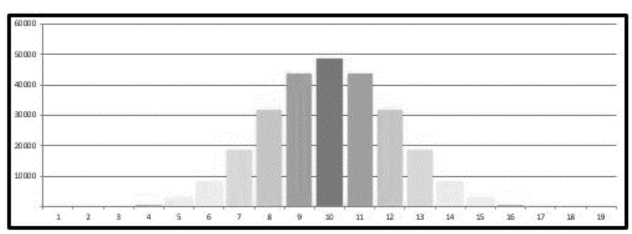
[xi] Unless you are entering or editing a rule, which spreadsheets usually call a formula or function they symbolize as $\mathbf{f_x}$, you don't see it because rules lie hidden inside the output cell.
[xii] Sum a column of numbers where we simply fill in the cell addresses between the parentheses.

complex and sophisticated models surprisingly easily and quickly. For example, if you wanted to create a monthly budget you can copy the column for the first month into all of the rest. By doing so you are automatically copying its formulas like sum or average or percentage and with relative addressing these rules apply to the right data in each column. If some of the formulas refer to a single special cell, say one with your yearly salary, you can keep that reference in place by using its absolute address. This ease of model building and editing makes “What if…” questioning and experimentation easy and flexible.

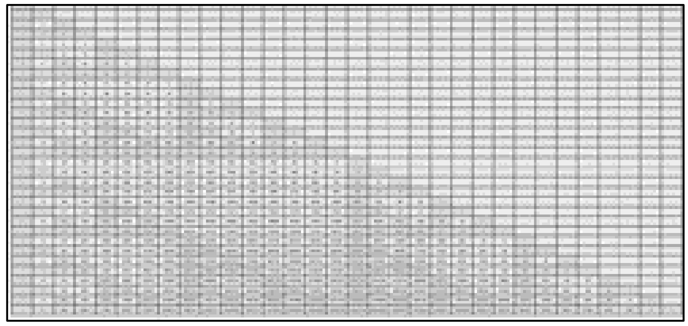
Spreadsheets let us edit and UNDO. Since undo[14] is a simple idea available in almost every computer program, it is easy to overlook its great potential for enabling learning. Undo gives students the power to play and replay, to experiment and re-experiment, to make and remake. This is something many students need to learn to do and practice. Erasing paper is not easy on homework or tests, and many students worry about neatness, so few students are encouraged in today’s classrooms to experiment, to play. Yet, we hear over and over from business that this willingness to try and fail, to take risks, to be wrong is a critical skill. Whether or not “fail fast, fail often” is actually good business practice, a willingness to take some risks is, and spreadsheets can help students learn this skill. **Undo** may therefore be one of our most powerful tools for building problem solving and creativity.

Spreadsheets can make functions and “What if…” thinking concrete. One of the most common complaints about math education today is that it is abstract. Focused on symbols and symbol manipulation, it lacks the visual patterns that tie it to the things that students see and touch. Spreadsheets are uniquely positioned to connect the abstract to the concrete, the symbol to the visual, the problem to real data. Spreadsheets can automatically link formulas to tables, graphs, and images. These *linked representations* enable students with different learning modalities to understand and work with mathematical concepts. Spreadsheets also link discrete (concrete) and continuous (abstract) functions in an intuitive way by letting students graph tables as discrete points and then convert the graph to a continuous line. And their natural linking through cell addressing lets us build amazing abstract patterns that we can explore concretely.





> *For example, pick a cell on a blank spreadsheet, put in a rule that adds the cell above it and the one to its left. Copy that formula into a big area of the spreadsheet. Seed it by putting a 1 into the first cell. Go ahead and try it. This simplest of links blew me away when I first did it, and it continues to thrill me with the amazing links across mathematics from probability to binomial expansion, from number theory, to calculus, and more.*



This ability to link, lets us graph one of those rows and get a Bell curve, add the numbers in a row and get a power of 2, follow a diagonal to get Fibonacci's sequence, or color the odd numbers to get a fractal pattern called Sierpinski. And this is just the beginning of the patterns found in one simple rule.

The Science of Patterns

In 1987, a seminal article appeared in the journal *Science* by Lynn Arthur Steen the prominent American mathematician and educator entitled "The Science of Patterns". It is highly unlikely that any article in the history of this prestigious journal had its title reprinted as often. In the article Steen suggests a new definition for mathematics:

> *The rapid growth of computing and applications has helped cross-fertilize the mathematical sciences, yielding an unprecedented abundance of new methods, theories, and models. Examples from statistical science, core mathematics, and applied mathematics illustrate these changes, which have both broadened and enriched the relation between mathematics and science. No longer just the study of number and space, mathematical science has become the science of patterns, with theory built on relations among patterns and on applications derived from the fit between pattern and observation.*[15][xiii]

His words decorate classrooms and beautify textbooks in an effort to persuade students that math really is beautiful, relevant, and exciting. They have not succeeded.[xiv] They have not changed the math students are supposed to learn, the way it is taught, or the standards set for it. If Steen's redefinition of mathematics as *the science of patterns* based on *technology* and *application* is to apply not just to math itself but to math education, then any new curriculum should be founded on:

- **Pattern** and patternmaking as the goal of the learning
- **Technology** as its central and essential tool
- **Application** to invite, entice, and engage students in the process

Reinventing Mathematics Education for the 21st Century

Over the past two years I have led a small team developing archetypes for such a curriculum. As I write this, we have nearly 100 spreadsheet lessons. We call them "Labs" because they are experiments using spreadsheets as laboratories. You can find them at www.whatifmath.org. They are **free**!

We expect most students to take about 20 minutes to complete one. But all are "What if…" lessons so students may spend more time as they enter "strange new worlds."[16] We offer these as models for we expect thousands of such lessons, enriched by a multitude of possible applications that engage students in a wide variety of different subjects and problems. *Liber abbaci* had more than 600 pages of business

[xiii] My underline.

[xiv] Only 1% of all U.S. college graduates get a bachelor degree in mathematics or statistics. And only 20% of entering college students are "academically ready for the rigor of the first-year college course…if they are planning a STEM career." *Education Week Blog 11.11.15*

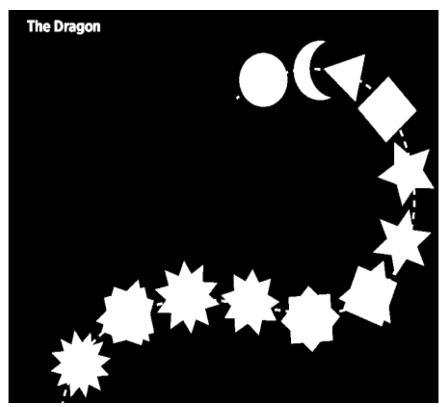
problems to give merchants practice using Arabic mathematics and paper technology. We imagine a 21st century web-based equivalent. We have adapted the well understood case-study, problem-based learning, model commonly used in business schools and higher education[xv] for K-12 providing choice, creativity, and challenge as essential features. We expect students to work in teams, enhanced by Internet communication and resources, with powerful imaginative technologies like spreadsheets, and take charge of their work and their learning. We divide the curriculum into three broad and flexible stages, based on child and patternmaking development.

Stage 1—3 Years to 3rd Grade

Building numbersense through counting using patterns, actions, physical objects, and spreadsheets as physical objects.

> *Studies find that the mathematics knowledge acquired in early childhood and early elementary grades is a critical foundation for long-term student success. A child's math ability when he or she enters school has proved a better predictor of academic achievement, high school graduation, and college attendance than any other early childhood skill. Early mathematics competency even predicts later reading achievement better than early literacy skills. Finally, high-quality early mathematics instruction supports later learning of the science, technology, engineering, and mathematics (STEM) skills U.S. employers are demanding.*
>
> National Governors Association Paper 2014

The data is crystal clear. Success in school is highly correlated with numbersense. A child who cannot count things with one-to-one correspondence cannot use number. A child who does not know that 18 is greater than 15 cannot write a number sentence. A child who cannot count by 10's will not understand place value. A child who can't count forwards will not recognize addition or count backwards to make sense of subtraction. Without the ability to count-by a child will find multiplication and division a mystery and the times table very hard to memorize. Can't count by 5's, forget about telling time on an analog clock or making change.



Sidewalk Math Dragon Pattern
Count Corners

Numbersense takes practice with patterns, lots of practice with patterns, especially the counting patterns, and it also takes brain development. Many children come to school today without numbersense. Some develop more slowly, but most lack practice with patterns, especially counting patterns. Our job, in 3 to 3, is not just to help and encourage students to build numbersense patterns; it is to be very sure we give students the time to develop their brains to be ready for such construction. It is of little value, most likely of negative value, to force children without numbersense to practice paper algorithms. A primary goal of our 21st century Stage 1 curriculum must be to ensure that every child has the time and the practice to develop numbersense.

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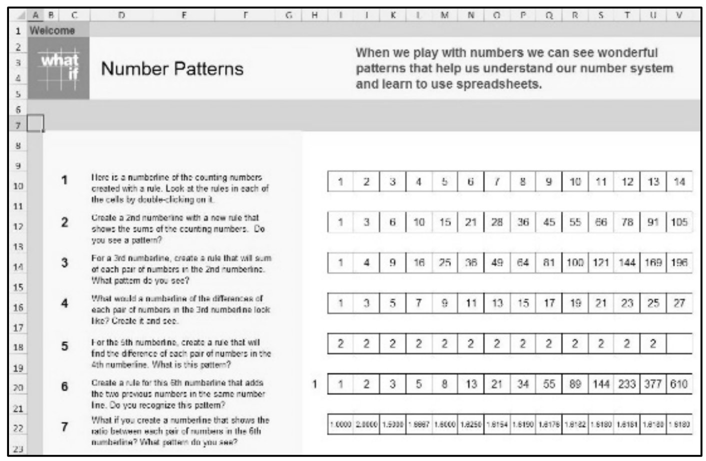
xv Harvard Business School, Law School, and Medical School among others base their curriculum on this model.

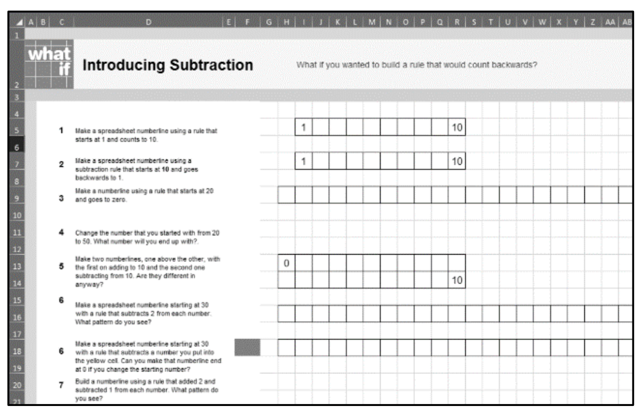
Sidewalk Math[xvi] patterns painted on sidewalks or playgrounds give kids a chance to learn numbersense patterns by play and reminds parents and caregivers that they should be counting with their children. These beautiful patterns, available at www.sidewalkmath.org, invite children to make counting a playful, action-filled, community activity that can enable every child to come to school seeing, building, and enjoying the patterns in numbers.

Spreadsheets should be introduced as soon as a child wants to play with them. [xvii] We developed early Labs for students to make numberlines, introducing them to cells, rules, inputs and outputs. They put a number into a cell and make a rule in the next cell to add 1 to that number, they copy and paste that rule into more cells in the row to make a numberline. These numberlines are, for all intents and purposes, physical entities, containers, that students create, change, color, or add borders. They can change the mathematical values and patterns in these numberlines by starting with a different number or by changing the rule, turning their spreadsheet into a "manipulative" that automatically counts and calculates!

From numberlines, student build 100's tables, times tables, rules that add or subtract, rules that count by 10's, or rules to count by the numbers from 1 to 12 to build multiplication tables. They go on to see the patterns in these tables, to find rectangular numbers, square numbers, triangular numbers, prime numbers, and to explore them as patterns in numberlines and tables asking "What if…" questions like "How many of the numbers in a times table are odd?" "What does the graph of the triangular numbers look like?" "What if I multiply the numbers in the opposite corners of any rectangle I draw on a times table, are those products always equal to each other, and why?"



As an example, imagine learning subtraction using a spreadsheet. Today, teachers generally write a number sentence like 7 – 3 = ____ while showing students 7 objects and 3 objects and tell them "seven take away three is four." But the students ask themselves, "What happened to those other three objects the – 3 that I was supposed to take away?" "I took three objects away from the seven but I still have those other three left over from the number sentence." And



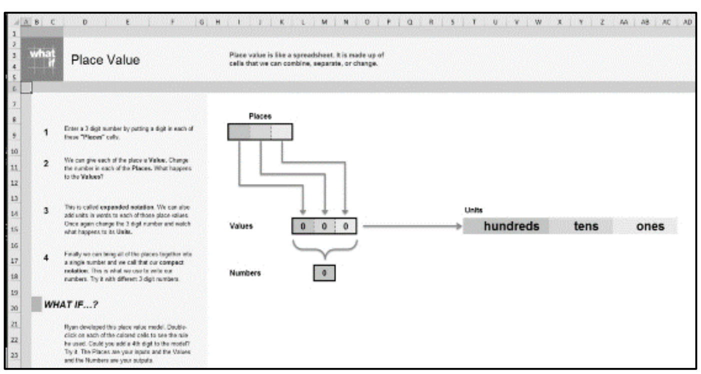
[xvi] Sidewalk Math, invented by the author, supported by and available from Lesley University at sidewalkmath.org, is a collection of hopscotch-like patterns on sidewalks and playgrounds that use art and play to invite children to count in all of these ways and thus build numbersense.

[xvii] We often under estimate the ability of young children to use technology. Many come to school with rich experience using computers, phones, and tablets. The need to ensure equity for our preschoolers in technology experience may be one of the most important educational problems our nation and institutions must solve.

that is just the beginning of the confusion the subtraction paper algorithm can cause. If students were to learn subtraction using a spreadsheet, they could build a number line using a subtraction rule after making spreadsheet numberlines with addition rules, introducing subtraction as the inverse of addition. The input would be a number on the right and the output would be the number next to it on the left. The rule would be: subtract 1. They would learn subtraction as a way to count backward and picture it as a numberline and a simple functional rule, the inverse of addition. They would picture subtraction as a counting activity, a dynamic creative action that they can use to build and play with numberlines and explore number patterns. They could then change the rule to subtract 2, 3, 4, etc. to practice the facts and go on to build a subtraction table to include all the facts.

12	12	24	36	48	60	72	84	96	108	120	132	144
11	11	22	33	44	55	66	77	88	99	110	121	132
10	10	20	30	40	50	60	70	80	90	100	110	120
9	9	18	27	36	45	54	63	72	81	90	99	108
8	8	16	24	32	40	48	56	64	72	80	88	96
7	7	14	21	28	35	42	49	56	63	70	77	84
6	6	12	18	24	30	36	42	48	54	60	66	72
5	5	10	15	20	25	30	35	40	45	50	55	60
4	4	8	12	16	20	24	28	32	36	40	44	48
3	3	6	9	12	15	18	21	24	27	30	33	36
2	2	4	6	8	10	12	14	16	18	20	22	24
1	1	2	3	4	5	6	7	8	9	10	11	12
*	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**

Though such a foundation may appear revolutionary in light of today's curriculum, it would be entirely natural to young students. For our kids should come to school steeped in a world of patterns and patternmaking[xviii] gained from singing songs, the same songs endlessly; playing games, repeatedly; watching the same movies; drawing the same pictures; putting together the same puzzles; loving the same poems; dancing the same steps; and for the most part living in a world of patterns they make or inherit. We are all patternmakers. We make them to choose, order, store, and retrieve the wealth of experience our senses constantly feed our minds.[xix] A mathematics curriculum built as a science of patterns using technology and functional thinking would naturally focus on patterns with broad applicability. For example, when students construct a times table on a spreadsheet, the vertical axis should go up instead of down to match to the Cartesian coordinate system connecting the times table to ordered pairs, graphs, slope, negative numbers, and more. In such a curriculum we would make linking (formulas to tables to graphs) so natural, standard practice and not a special exercise. We focus on patterns like symmetry so students see the times table as a reflection around its diagonal realizing they only need memorize half the products. Functional thinking would always be the focus: "What is the input?" "What is the output?" "What is the rule?" "What pattern do you see?" and of course "What if... you change this or do that?" in physical activities, headmath practice, and spreadsheet Labs for students to practice:



Place Value

[xviii] This may not be true for children who grow up in the chaos of poverty or war.

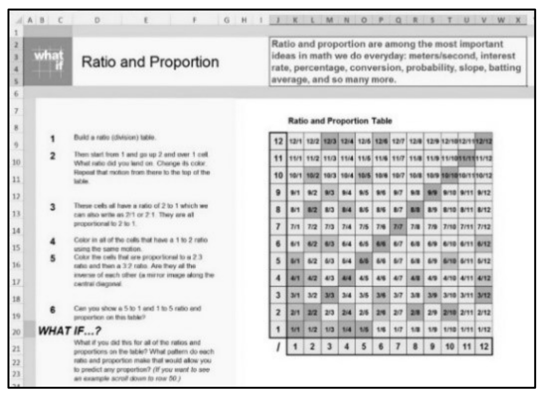
[xix] To further explore patterns and patternmaking see my book, *Elegantly Simple: On the Future of Knowledge*, 2013, a study and theory of human patterns and patternmaking featuring a comprehensive pattern to knowledge.

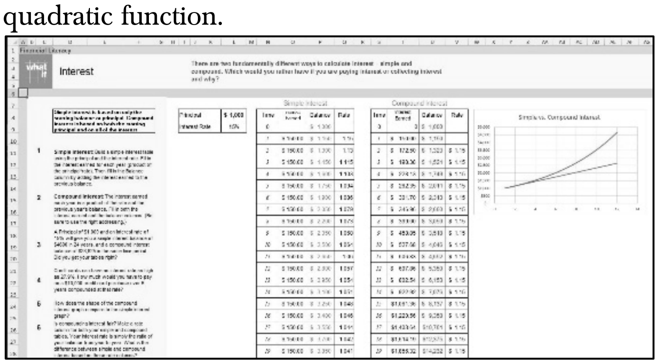
- **Counting** the whole numbers forward and backward to 100. Big numbers are fun. Children should count at every opportunity and ask, "What patterns do you see?"
- **Counting objects** to link number and quantity, do this at every opportunity.
- **Counting-on** to make a rule that makes new numbers by adding 1 to the starting number. Build number lines on spreadsheets. Ask "What if I change the starting number?" Use counting-on as the building block of addition and going backward for subtraction even to negative numbers.
- **Counting-by 10's** to make a spreadsheet rule that will build a hundreds table and place value.
- **Counting-by 1 to 12** to make rules that will build multiplication patterns and the times table. Ask "How many of the products in a 12 *12 times table are odd numbers. Use Counting-by and rectangles as the building blocks of multiplication.
- **Counting by time** is counting by 5's
- **Counting corners** to connect shape and number and to see numbers as physical patterns as the Greeks did. Students should count sides, corners, look for patterns: even numbers form rectangles, square numbers form squares, and triangular numbers (the sum of the counting numbers) make a triangular pattern that connects addition and multiplication.
- **Counting to measure** time as a repeating sequence, distance as a line, area as a product to build the mental patterns so necessary to take the measures of things.

Stage 2 – 4th – 7th Grades

Developing the patterns leading to abstract thinking, big patterns like ratio, and link them to concrete experiences and patterns ("recognizable things."[17])

Ratios are ubiquitous, central to the problem solving activities found in business and life: rate, interest, percentage, conversion, fractions, linear functions, slope, and many more. Ratios, the quotient of two numbers or more generally two quantities, are functions. Like most functions they can be represented as a formula, table, graph, or even a physical image. They are so important that we often represent a function, like 5:8, as a single number: a fraction 5/8, decimal, 0.625, percentage 62.5%, or batting average .625. By starting with ratio, students will learn to see these numbers as a shorthand for an ordered pair, or more accurately a table of ordered pairs (5,8), (10,16), (15,24) ... that can be represented as a line with a slope of less than 45°, a graph, a linear function $f(x) = \frac{5}{8}x + b$, a trig function the sin(.625)=0.585, and even the derivative of a quadratic function.



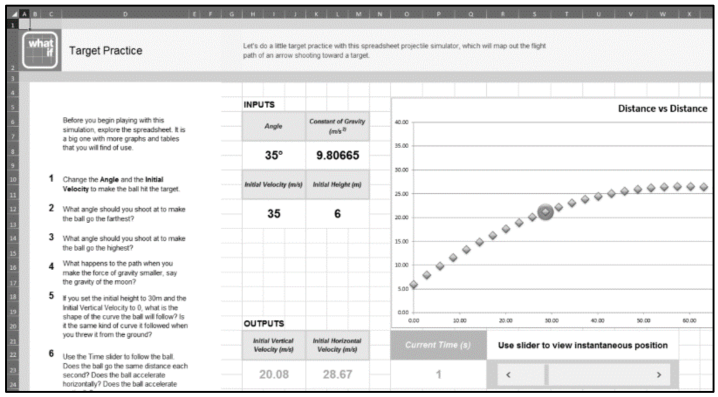


In the "traditional"[xx] curriculum most of these siloed forms of ratio are introduced as separate patterns with the focus on their paper algorithms. When we develop ratio as a general pattern using functions and functional thinking, we make many of these ideas much easier for students to understand. For example,

[xx] Problems involving work, interest, investment, tolerance, commission mixture, motion, taxi fares, etc.

conversion between fractions, decimals, percents becomes a trivial **Format** command. We don't have to differentiate ratios and proportions, for a ratio is always a table and each of its ordered pairs is a proportion. And since all ratio-based problems involve an input, output, and rules students learn to approach what today are separate ideas, as a single methodology. This power of technology and functional thinking enables younger students to easily conquer what used to seem to be difficult stuff.

- **Ratios as functions** – inputs, outputs, rules in tables, formulas, graphs, and objects
- **Ratios as division** – the most important operation
- **Ratios as percents** – standardized ratios
- **Ratios as numbers** – fractions and decimals turn ratios into numbers, fractions are useful in headmath.
- **Ratios as rates** – Scores, motion, etc.
- **Ratios and variation** – direct and indirect
- **Ratios as patterns** – Golden ratio, Pascal's triangle, Fibonacci sequence, convergent/divergent series
- **Ratio as linear functions** simple ratios often with an additive component.
- **Ratio as slope** – graphs of linear functions.



- **Ratio as rational and irrational numbers and geometric shapes** – Circles and πangles, right triangles are ratios. Yes, π is a ratio.
- **Ratio as probability** – Chances, random rules, central tendency, functions of multiple quantities
- **Ratios in business problems** – Simulations like Lemonade Stand, Pizza Parlor, interest rates,
- **Ratios in probability and Statistics problems** – Chances, averages, scores, etc.

Stage 3—8th Grade and Beyond

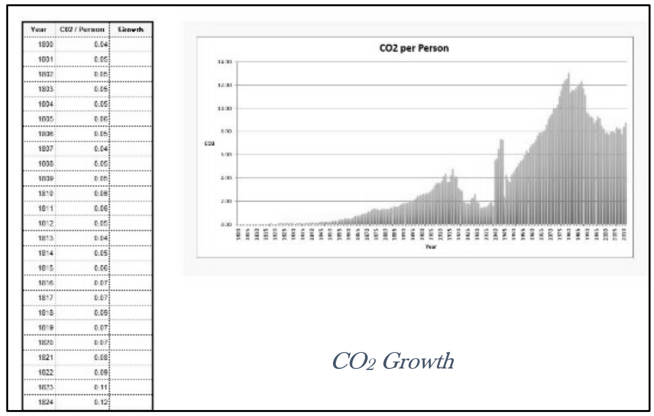
Learning to apply the patterns in mathematics to authentic problems involving rich data and abstract thinking.

"Relate the school to life, and all studies are of necessity correlated." John Dewey[xxi]

We have built three types of Labs for students at this stage: 1) Labs that focus on working and finding patterns in significant data, the kinds of problems and cases business school students would find relevant; 2) Labs involving financial reasoning which we believe is critical learning for every student; 3) Labs involving fascinating functions and functional patterns. All of these lessons would involve students in functional thinking, in coding, and in working, sharing, and presenting to others.

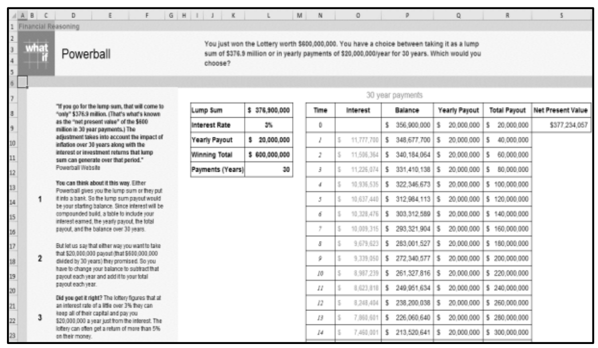
[xxi] John Dewey, *The School and Society*, 1899

We have built Labs to get students solving authentic problems involving data on climate change. One looks at data from the U.S. on CO_2 emissions since 1800. We ask students to explore that data, to build graphs and tables to answer questions about the data. Then we ask our critical question, "What if you had to show that CO_2 emissions were a serious problem, what would you present and argue." Rich problem solving enabled by functional thinking, the problem solving now standard practice in business, industry, and technology based on sophisticated models, abundant data, and complex interactions, requires students not only to know how to navigate but to know how to weed out the irrelevant, to simplify the data, and to clearly present the results.



CO_2 Growth

Every student needs the tools to work with money at home and at work, and since spreadsheets are widely used by individuals and businesses for their financial planning, tracking, and decision making, we believe that *financial reasoning* must be a focus of the Stage 3 curriculum. We have developed Labs to enable students to learn to manage their personal finances, understand interest rates, know a good deal, make and track budgets, recognize lotteries for what they are, and deal with business issues like margin vs. markup.

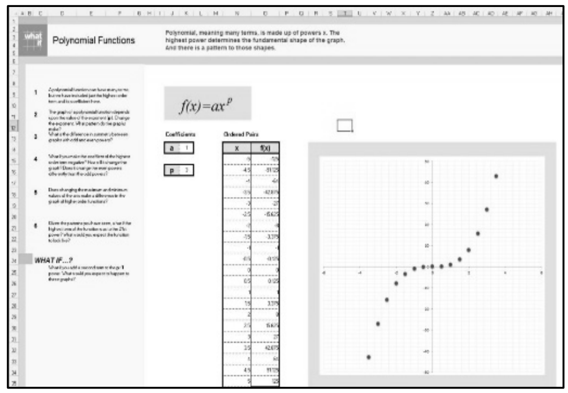


Powerball

Functions are transformations; even simple ones can be fascinating, adding new dimensions to old subjects. I found this out while I was looking for a function to use to show recursion. I found it in a video on the Web. $x = \frac{1}{x+2}$, the video explained was recursive because the variable appeared on both sides of the equation. This equation, despite being typical student nightmare, appeared interesting though I am sure that the video made it seem even more complex.

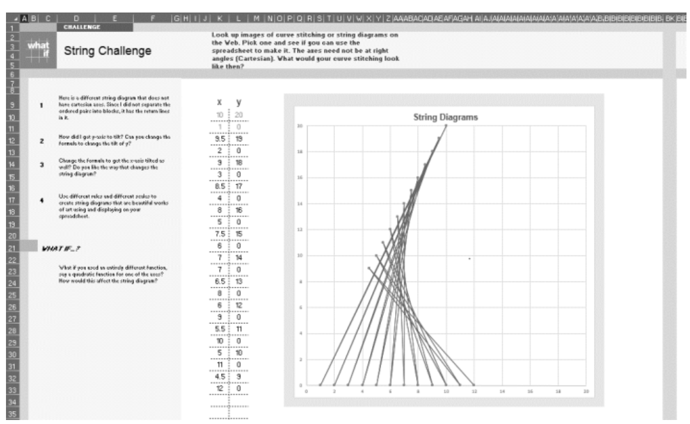
I brought it to Peter who suggested we use our standard functional thinking methodology, by building a table in which each succeeding row used the value of x generated by the previous row (recursion) instead of turning off automatic recalculation as the video suggested. Throw in an initial value for x and run the table far enough and it converges to -0.414. Great, we got a solution to what on the surface seemed like a nasty equation. But take another look at this equation. Do a little bit of cross multiplying algebra to get rid of denominators and it turns into $x^2 + 2x = 1$ or in the more conventional form, $x^2 + 2x - 1 = 0$. This nasty looking equation is a simple quadratic that most students would immediately say it is not factorable and would have to be solved by applying the quadratic formula. We were elated to find that our solution was indeed one of them. Then the big question "Where was the other solution?" Most quadratic equations have two solutions.

Peter and I were captivated and started asking everyone we could find. Finally, my nephew said, just cross multiply, $x = \frac{1}{x+2}$ $x + 2 = \frac{1}{x}$ or $x = \frac{1}{x} - 2$ [xxii] to switch the denominators, then build a recursion table to get the second solution -2.414. I felt that we had discovered a new way to solve a quadratic equation. We generalized it as $ax^2 + bx + c = 0$ and could easily build a recursion system that would enable us to solve any quadratic equation in this way without knowing or remembering the quadratic formula. I don't know if others had come up with this same idea, and to tell you the truth, I don't care. We were part of making a new mathematical discovery of our own, and the thrill of discovery, of being creative remains in me today. Functions and functional thinking enabled us to view even basic algebra in new ways and enable students, and we count ourselves as students, to play, experiment, and explore mathematics finding things that excite and engage us.



Power Functions

- **Coding Functions** – Model building, if…then, recursion, logical formulas, If, and, or, true, false, etc.
- **Attributes of Functions** – Evaluation, domain, range, intervals, inverse, maximum, minimum, point of inflection, slope, area, vertical line
- **Families of Functions** – Non-linear functions, operations, composition
- **Geometry of Functions** – transformations
- **Functions and Calculus** – derivative as slope, integral as area
- **Probability and Statistical Functions** – randomness, probabilities, data representations, central tendency, deviation,
- **Financial Reasoning** – Personal and small business finances, financial models,
- **STEM** – Modeling, experimenting, data representation, large data samples
- **Math and Art –** Spreadsheet design, data display graphs and charts, interactive spreadsheets, representing data, modeling



This reinvention of the mathematics curriculum using problem-based-learning, proven highly successful in graduate education, is now made possible in K-12 by technology (spreadsheets and Internet). It replaces the rigid staircase with its paper algorithm barriers with an elevator that enables every student to develop at her or his own rate, to choose from thousands of lessons, developed by a wide variety of creative people to provide the kinds of small and large case studies that enable every

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[xxii] If we were to write these equations in paper form, we would likely use subscripts to make it perfectly clear that x actually represents two different values in each of the equations where **n** is the nth term and **n+1** is the next term after that. They would then look like this: $x_{n+1} = \frac{1}{x_n + 1}$ and $x_{n+1} = \frac{1}{x_n} - 2$.

student to find appealing and interesting problems. It enables students to work together in groups, to share, to help each other, and to find the support they need on the Web so that the demands on teachers are substantially reduced. It takes portfolio and juried assessment from the arts to the rest of the curriculum. It can challenge every student to grow as rapidly as they can, to make learning their own creative experience, and to develop the grit so crucial to building confidence and success. Finally, it enables students to practice with the tools and learn the patterns they will actually use as adults, to stifle once and for all the painful question, "Why do I have to learn this?" and engages students in learning that is a creative experience.

Some may wonder how such a radical reinvention of math education would affect preparation for college level courses (calculus, statistics or other STEM subjects). The College Board tells students that they need a solid background in functions as a pre-requisite to taking an AP calculus course.[xxiii] The research bears this out, for success on the PCA[xxiv] precalculus assessment focused on understanding functions is highly correlated with AP calculus test results.[xxv] We would certainly expect that students who practiced using functions and functional thinking from their earliest elementary years would do very well on calculus exams and courses. Likewise, a spreadsheet makes the use of statistics, graphing, and real world problem solving a normal activity that would similarly prepare students for college work. And for the 60% or more of community college students who fail to meet even the minimum math requirements for enrolling in a college level courses[xxvi], a new math curriculum and pedagogy focused on authentic problem solving not symbol manipulation, on multiple-linked-representations not just numbers and letters, and on creative functional thinking not paper algorithms can enable everyone to learn, understand, and use the mathematics they need for their career and life.

[xxiii] ...before studying calculus, students must be familiar with the properties of functions, the algebra of functions, and the graphs of functions." (College Board 2014)
[xxiv] Precalculus Concept Assessment Instrument (Carlson, Oehrtman & Engelke, 2010)
[xxv] Vrabel, Adam, Function Conceptions of AP Calculus Students, Lourdes University Thesis, 2014
[xxvi] http://ccrc.tc.columbia.edu/Community-College-FAQs.html and of these only a third succeed.

Learning Mathematics as a Creative Experience

That math education will take this form; I have no doubt. Leonardo's math reminds us that education must and will meet the demands of jobs, and CEO's have "identified creativity as the number-one 'leadership competency' of the future."[18] It is therefore the work of teachers to, in the words of the renowned educator Roger Brooks,[19] "Make your kids creative, that's the work of the college professor, of the 7th grade teacher, and of the teacher of 7 year-olds."

What may surprise us is the speed with which such revolutionary innovation can occur. Though schools seem to change at glacial speed; there are times when it has moved as fast as a charging river. I lived through such a change in the 1950's when desks were unbolted from floors. Mobile desks did not generally alter their orientation, but they dramatically changed their number. My 5th grade Chicago public school classroom had 51 students, three mobile desks added to the standard 48 attached to the floor. My 6th grade suburban classroom had only 25 students. Mobile desks take up more room, enabling teachers to bring a new level of interactive group pedagogy to their classrooms.

The ingredients are in place for a similar transformation in our classrooms. The technology—hardware, computers, tablets, even phones—is ubiquitous and cheap. The software—full powered spreadsheets—is free to students. The connectivity—high-speed internet access in schools and homes with FAQ's, YouTube videos, Facebook—provide plentiful opportunities for students to share their work and find support. Our vast high-stakes assessment system that seems to many to inhibit change can easily be converted to a change agent, by slightly alter its instructions to students to make it Open-Internet. Instead of blocking access to most of the Web, as we currently do, we would enable and indeed encourage students to search the Web, use tools found on the Web, and communicate using the Web.[xxvii] Naturally the test questions would have to change as would the curriculum, for if our tests are Open-Internet, then obviously most assignments must be as well if classroom are to prepare students for the tests that prepare them for the jobs.

To imagine these classrooms, I picture an office meeting where a manager brings a team together to creatively solve a problem. The manager would <u>not</u> say, "Put away your computers, your tablets, your phones, don't talk to each other, take out a blank piece of paper, and fill in your solution."[xxviii] If we want to prepare students as best as we can for the jobs of the future we know they will all need to become creative problem solvers—making patterns, working together, finding information on the Web, and using functional thinking to ask "What if…" and not "What is____?" They will have to learn mathematics as a creative experience.

[xxvii] Subject, of course, to the reasonable restrictions blocking porn, gaming, and other activities that might put a student into dangerous or compromising situations. We must remember that the purpose of schooling is not to protect students from dangers in their world, but to teach them how to protect themselves!

[xxviii] Some of you may think that this type session might sometimes be valuable as a change to prevent premature conclusions or to get some headmath going. And no doubt this should be in every manager's repertoire, but in general, modern managers should give their people the best tools to solve problems and not limit them. And we should teach our children those best methods we deeply believe in.

Notes

[1] I was introduced to Leonardo and *Liber abbaci* by Keith Devlin in his fine biography of Leonardo, *The Man of Numbers: Fibonacci's Arithmetic Revolution*, 2011, Walker & Co. Keith uses this unconventional spelling of Leonardo's title which I copy.
[2] It is not known who or what Leonardo was referring to.
[3] http://oll.libertyfund.org/titles/753
[4] I am indebted to Professor Owen Gingrich for this insight.
[5] There were earlier images that we could view as graphs, perhaps charts would be a better word for them, that for example traced the path of a planet over time. (See Bruce S. Eastwood, Astronomy and Optics from Pliny to Descartes, Wariorum Reprints, London 1989, p. 278). But I do not consider these to be conceptual representations.
[6] W. W. Rouse Ball, A Short Account of the History of Mathematics, http://etc.usf.edu/lit2go/218/a-short-account-of-the-history-of-mathematics/5529/bonaventura-cavalieri/
[7] Isaac Newton, *Tractatus de Methodis Serierum et Fluxionum* (1671; "Treatise on the Method of Series and Fluxions") and from Frank Wilczek, A Beautiful Question: Finding Nature's Deep Design, Penguin Press, 2015, p. 383
[8] Jere Confrey, David Dennis, Functions of a Curve: Leibniz's Original Notion of Functions and its Meaning for the Parabola, http://poncelet.math.nthu.edu.tw/disk5/js/linkage/m7.pdf
[9] Ada Lovelace 1842/43[9]
[10] I recall that in the 1960's and '70's a mathematics/science program for primary grade students by Robert Karplus taught math with functions using a similar diagram.
[11] This graph does not even contain the latest test results which came out today – Oct 27, 2015 which show once again, no change, no significant change at all, a flat line!
[12] Here is how I get 20%. Of the students in any given year only 2/3rds go on to higher education from high school. I am sure that if you are good in math you see yourself as a good learner and you at least start some college work. Of that 2/3rds over 1/3 (36% by recent estimates, fail their incoming college readiness test in math, they did not learn arithmetic or high school algebra well enough to go on without remedial help. This leaves us with about 40% of our population who learned enough math in K-12 to go on to college work. Of that number, in my experience talking as a math teacher with many people I meet with a college education who tell me that they did not really get math, that they were able to pass the courses but not understand or be able to use it. In my experience they represent half of the college educated population and half of 40% is 20%. I think that this sad, sad number is actually high!
[13] Some versions of this picture use the word function to represent the rule inside the box. But a function includes not only the rule but the domain it applies to and the range it produces. So I like this diagram in which the function represents the whole.
[14] I am grateful to Steve Bayle for this insight
[15] Lynn Arthur Steen, Science, April 29, 1988
[16] From Star Trek's, opening monologue.
[17] Steve Bayle suggested this language as well
[18] The Innovators DNA, 2011, Jeff Dyer, Hal Gregerson, Clayton Christensen, Harvard Business Review Press p. 7
[19] Roger Brooks, President and Chief Executive Officer of Facing History and ourselves and former Dean of the Faculty and Chief Academic Officer at Connecticut College